IMSC 2058 Solution for Homework 3

1. Suppose that $\mathbb{R}^n$ is endowed with the usual metric, that is, for $x = (x_1, \cdots, x_n), x = (y_1, \cdots, y_n) \in \mathbb{R}^n, d(x,y) = \sqrt{\sum_{k=1}^{n}(x_k - y_k)^2}$. Show that if Y is a proper vector subspace of $\mathbb{R}^n$, that is $Y \subsetneq \mathbb{R}^n$, then Y is nowhere dense.

   Solution

   First, note that any linear subspace of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$. Hence, $\overline{Y} = Y$. We denote the interior of Y by $\text{int}(Y)$. To show Y is nowhere dense is equivalent to show $\text{int}(Y)$ is empty.

   Assume $\text{int}(Y)$ is nonempty. Then there exists some $y \in Y$ and $r > 0$ such that the open ball $B(y,r) = \{x \in \mathbb{R}^n : d(x,y) < r\} \subset Y$.

   Let $v \in \mathbb{R}^n$ such that $d(0,v) < r$. Since

   $$d(y, y+v) = d(0,v) < r$$

   then $y+v \in B(y,r)$ and $y+v \in Y$. Also note that Y is a vector space, then $v = (y+v) - y$, which implies $v \in Y$. Thus, $B(0,r) \subset Y$.

   Let $w \in \mathbb{R}^n$ be arbitrary with $\| w \| > 0$. Let $t = \frac{r}{2\|w\|} > 0$. Then

   $$\| tw \| = \frac{r}{2} < r$$

   which implies $tw \in B(0,r) \subset Y$

   Since Y is a subspace, it is closed under scalar multiplication, so $w = \frac{1}{t}(tw) \in Y$. This implies $Y = \mathbb{R}^n$, contradicting the assumption that Y is proper. Therefore, $\text{int}(Y) = \emptyset$, and Y is nowhere dense.

2. Let A be a subset of X.

   (a) Show that if X is complete, then A is complete if and only if A is closed in X.

   (b) Show that if A is complete, then A is closed in X.

   Solution

   (a) Let A be a closed subset of a complete metric space X. Let (x_n) be a Cauchy sequence in A. Then (x_n) is also a Cauchy sequence in X. Since X is complete, (x_n) is convergent. Let $L := \lim_n x_n$. As (x_n) is a sequence in the closed set A, its limit L is also in A. Therefore A is complete.

   On the other hand, if A is complete, then A is closed in X by (b).

   (b) Let A be a complete subset of X. Here X may or may not be complete. Suppose (x_n) is a sequence in A converging to a limit K in X. We need to show that $K \in A$ As (x_n) is a convergent sequence, then it is also a Cauchy sequence. Since A is complete, (x_n) converges to a limit K'. in A. Now the uniqueness of limit implies that $K = K'$ Therefore A is closed in X.